PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
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The use, misuse and abuse of mathematics in finance

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THE ROYAL SOCIETY

The use, misuse and abuse of mathematics in finance

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The once 'gentlemanly' business of finance has become a game for 'players'. These players are increasingly technically sophisticated, typically having PhDs in a numerate discipline. The roots of this transformation have their foundation in the 1970s. Since then the financial world has become more and more complex. Unfortunately, as the mathematics of finance reaches higher levels so the level of common sense seems to drop. There have been some well-publicized cases of large losses sustained by companies because of their lack of understanding of financial instruments. In this article we look at the history of financial modelling, the current state of the subject and possible future directions. It is clear that a major rethink is desperately required if the world is to avoid a mathematician-led market meltdown.

Keywords: mathematical finance; derivatives; risk management

1. Introduction

In 1973 Fischer Black and Myron Scholes published a paper on the pricing of options and corporate liabilities in the *Journal of Political Economy* (Black & Scholes 1973) and Robert Merton published a paper on the theory of option pricing in the *Bell Journal of Economics and Management Science* (Merton 1973). It was not easy getting these papers published; some of the ideas they contained conflicted with the then current and commonly held beliefs about the pricing of financial instruments. Nevertheless, just over 20 years later Scholes and Merton received the Nobel prize for economics for their work, Fischer Black having died a couple of years earlier.

In the quarter of a century since the publication of the Black–Scholes model, the financial world has grown enormously in both dollar size and in sophistication. The notional value of complex derivative products is measured in *trillions* of dollars. Individual traders, usually in their early twenties, control contracts worth hundreds of millions of dollars and command correspondingly astronomical salaries.

Once upon a time these traders would have studied History at Oxford and found a job via the old-boy network. Then in the 1980s it became fashionable for banks to hire so called 'East End barrow boys' without a university education, all that was required of a trader was gut instinct and bravado. But lately, only those with PhDs in mathematics or physics are considered suitable to master the complexities of the financial markets.

It would be pleasing for an academic working in financial modelling to see that advanced mathematics is finding a role in such an important global business. If only it were that simple. As fast as finance theory advances so does its misuse and abuse. We will start by looking at one example that illustrates this problem, the Procter and Gamble fiasco in which a derivatives transaction seriously backfired.

2. An example of how it all went wrong

Proctor and Gamble (P&G) is a major multinational company manufacturing beauty and health-care products, food and beverages, and laundry and cleaning products. They have a large exposure to interest rates and to exchange rates. To reduce this exposure they use interest rate and currency swaps, moderately sophisticated financial instruments. During the late 1980s and early 1990s, P&G had very successfully hedged their exposure, but had increasingly used financial instruments to speculate on the direction of interest rates. Again, they were very successful in this, making profits in addition to their 'proper' business. But no one can predict the market for ever.

In late 1993 P&G wanted to enter into a swap from a fixed rate of interest to a variable or floating rate, having the view that rates then low would remain low. A very basic form of swap (so-called vanilla) would be fine as long as rates did not rise, but what if they did? Bankers Trust (BT), the counterparty to the deal, suggested some modifications to the swap that satisfied P&G's concerns.

The deal, struck on 2 November 1993, was a five-year swap on a notional \$200 million. It contained something a little out of the ordinary. The deal went like this. BT pays P&G a fixed rate of interest on the \$200 million for five years. In return P&G pays BT a fixed rate for the first six months, thereafter a rate defined by

$$r_{\mathrm{C}} - 0.0075 + 0.01 \times \max\left(\frac{98.5}{5.78}Y_5 - P_{30}, 0\right), \tag{2.1}$$

where r_{C} was the rate on P&G's own corporate bonds, Y_5 the five-year Treasury yield and the price of the 30-year Treasury bond.† The Treasury yield and price would be known at the time of the first payment, 2 May 1994, at which time it would be fixed in the formula. In other words, the yield and price pertaining on that date would be locked in for the remaining four and a half years.

The best that P&G could achieve from this deal would be for rates to stay near the level of November 1993 for just a few more months, in which case they would benefit by

$$0.0075 \times \$200\ \text{million} \times 5 = \$7.5\ \text{million}.$$

Five- and 30-year rates had been falling fairly steadily for the whole of the 1990s so far, see figure 1; perhaps they would continue to do so, matching the stability of the short-term yields. However, if rates were to rise between November and May . . . .

Expression (2.1) increases as the five-year yield increases and decreases if the 30-year bond rises in value. But, of course, if the 30-year yield rises, the bond price falls and (2.1) increases. Although there is some small exposure to the slope of the yield curve, the dominant effect is due to the level of the yield curve.

In November 1993 the 6.25% coupon bond maturing in August 2023 had a price of about 103.02, corresponding to a yield of *ca.* 5.97%. The five-year rate was *ca.* 4.95%. With those values expression (2.1) was comfortably the required $r_{\mathrm{C}} - 0.0075$. However, rates rose at the beginning of 1994 and the potential \$7.5 million was not

† A bond is a contract that pays fixed amounts of money, coupons, every six months, say, and then a large lump sum at the end of the term, the maturity date. Bonds have value and can be bought and sold; they each have a price. The yield on a bond is the equivalent rate of interest that you would need to receive if you simply invested the bond's price in a risk-free bank account. Bond prices, and thus yields, are always changing. As yields rise, so bond prices fall and vice versa.

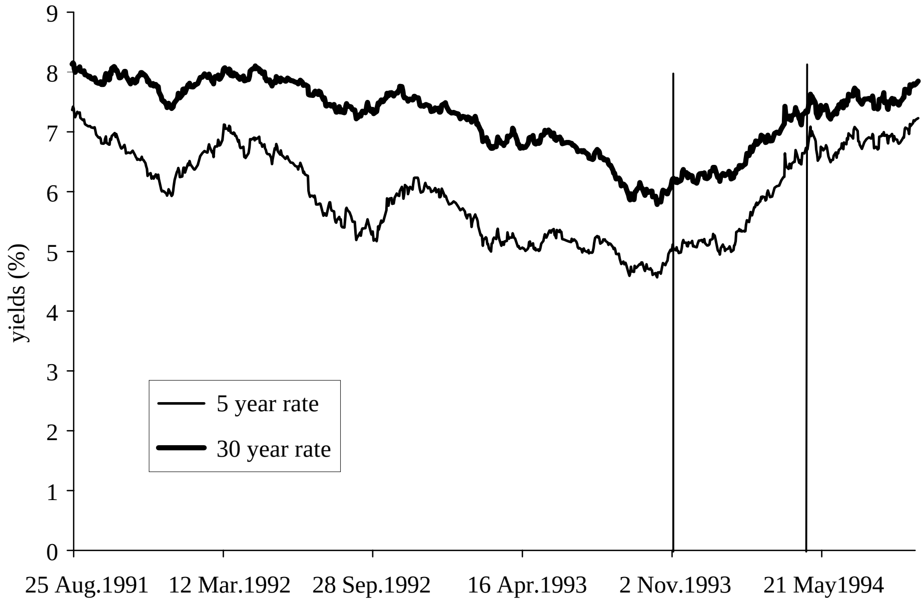

Figure 1. US interest rates in the early 1990s.

realized; instead P&G lost close to \$200 million. Look at figure 1: could the timing have been any worse?! Subsequently, P&G sued BT on the grounds that they failed to disclose pertinent information. The case was settled out of court.

The following was taken from P&G Corporate News Releases.

> P&G Settles Derivatives Lawsuit With Bankers Trust (9 May 1996)
>
> CINCINNATI, May 9 1996—-The Procter & Gamble Company today reached an agreement to settle its lawsuit against Bankers Trust. The suit involves two derivative contracts on which Bankers Trust claimed P&G owed approximately \$200 million. Under the terms of the agreement, P&G will absorb \$35 million of the amount in dispute, and Bankers Trust will absorb the rest, or about 83% of the total.
>
> 'We are pleased with the settlement and are glad to have this issue resolved,' said John E. Pepper, P&G chairman and chief executive.

It is not difficult to work out the potential losses *a priori* from a shift in the yield curve. This is done in table 1 assuming a parallel shift. P&G start to lose out after about a 0.7% rise in interest rates. Thereafter they lose *ca.* \$2.3 million per hundredth of a per cent. On 2 May 1994 the five-year and 30-year rates were 6.687% and 7.328% respectively, an average rate rise of over 1.5%.

Figure 2 shows the distribution of changes in US five-year rates over a six-month period during the 10 years prior to November 1993, data readily available at the time that the contract was signed. These historical data suggest that there is a 14% chance of rates rising more than the 0.7% at which P&G start to lose out (the black bars in the figure). There is a 3% chance of a 1.5% or worse rise. Using these data to calculate the expected profit over the five-year period, one finds that it is

Table 1. *Effect of parallel shift in yield curve on P&G's losses*

parallel shift (%)	0.0	0.5	1.0	1.5	2.0
five-year yield (%)	4.95	5.45	5.95	6.45	6.95
price of 30-year bond	103.02	97.77	93.04	88.74	84.82
30-year yield (%)	5.97	6.47	6.97	7.47	7.97
total loss over 4.5 years ($m)	0	0	75	190	302

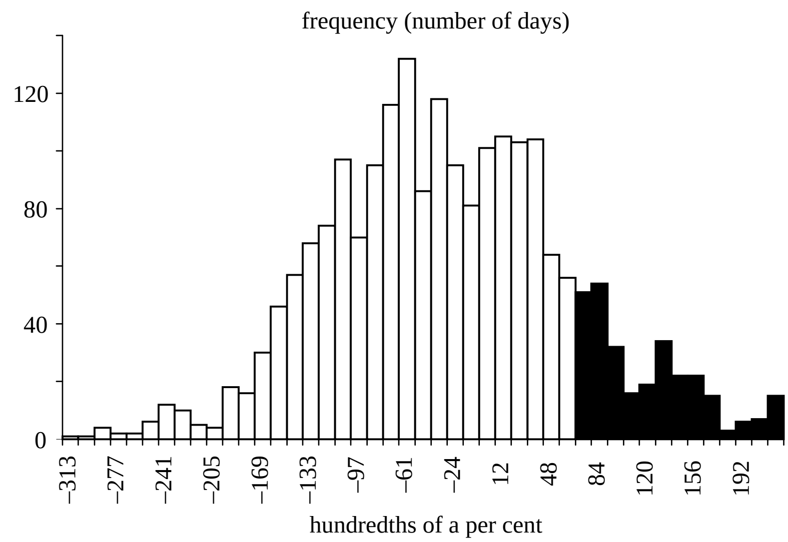

Figure 2. Distribution of changes in US five-year rates over a six-month period covering the 10 years prior to November 1993.

−$8.7 million, rather than the hoped for +$7.5 million.† Did P&G or BT do such a simple calculation? Although the financial instrument that caused the problem could be modelled using very high-level mathematics, there was an obvious role for a back-of-an-envelope calculation.

3. The Black–Scholes–Merton theory

Before we look at the role of mathematics in the development of the financial markets, we need to know a small amount about some of the financial products that are available in the market. We will only talk about some of the simplest instruments, using them as a nail on which to hang the ideas and concepts; nowadays there are many, many more products around, some of great complexity.

A stock (or share or equity) is a piece of a company. It has a value, depending supposedly on the value of the firm, but more immediately on the *perceived* value of the firm. If one believes that the company is undervalued by the market, one can invest in the company by buying some stock. If the stock rises in price, a profit is made. If one is wrong and the stock price falls, then one makes a loss. What if you

† The figure and the calculations use overlapping data. The same (actually, slightly worse for P&G) results are found if non-overlapping data are used.

believe that the stock is going to rise in value, but want to protect yourself against being wrong? Is there such a thing as insurance? In the financial markets there are contracts called *options* that have the required properties. A call option is the right to buy an asset at some time in the future, the expiry date, for a specified amount, the strike price.

Example. You believe that stock in company XYZ is going to rise significantly over the next three months. The stock price is currently £1.33 but you confidently expect a rise to the region of £1.65. If you buy the stock and you are right, then you make a profit of

$$\frac{1.65 - 1.33}{1.33} \times 100 = 24\%.$$

On the other hand, a call option with three months to expiry and a strike price of £1.40 may cost something like £0.04. The profit you now make if you are right is now

$$\frac{1.65 - 1.40}{0.04} \times 100 = 625\%.$$

This is a significantly greater return than if you simply bought the equity in XYZ. However, if you are wrong and the stock does not even rise above £1.40, the strike price, you will lose everything; the option expires worthless.

You would buy a call option if you think that the stock is going to rise in value but want downside protection. Similarly, a put option is the right to sell the stock on a prescribed date for a given amount.

Much finance research in the 1970s concentrated on the valuation of options assuming a specific model for the underlying asset, the stock. In 1973 Fischer Black and Myron Scholes and Robert Merton published their papers. Some of the ideas contained in those papers dated back to the early part of the 20th century, such as the representation of the random movement of stock prices as Brownian motion. Bachelier had written his thesis on option pricing in 1900. He had even gone so far as to present a theoretical model for the value of an option based on expectations; stock price movements are random but what happens on average?

Between Bachelier and Black–Scholes–Merton nothing much happened in the theory of pricing options. Bachelier's equity model using asset price changes drawn from a normal distribution was modified in the 1950s to normally distributed asset price returns (asset price changes divided by the asset price). This was the lognormal random walk for equity prices. But Black, Scholes and Merton made some insights that were to fundamentally change the financial world and in a very short time make finance a seriously mathematical subject.

Insight 1. Clearly, the value of a call option depends on the value of the underlying asset. If the strike price is fixed, then a rise in the value of the underlying asset will be accompanied by a rise in the value of the option. This is because the asset is more likely to end up at expiry at a higher value, giving the option greater value at expiry and greater value now. If the stock price falls, the option price will fall. Another way of saying this is that the call option changes and the asset price changes are correlated, positively correlated. A put option is negatively correlated with the underlying asset.

If the asset and a call option on it are correlated in this way, then a portfolio of one option and some quantity of the underlying asset will be insensitive to moves in

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

the underlying. As the asset price increases (or decreases) the option value increases (or decreases) and the sum of the two, suitably weighted, will remain constant. The number of assets one must sell to ensure this is called the 'delta', and is simply the correlation between the option and its asset. The process of eliminating portfolio sensitivity to changes in the underlying is called 'delta hedging' or 'dynamic hedging', hedging being the general term for reduction of variability or risk. In delta hedging, Black, Scholes and Merton had shown how to eliminate all variability by constructing a very special portfolio. In financial terms variability is seen as a measure of risk. Black, Scholes and Merton had eliminated risk.

Insight 2. They had constructed a portfolio that was totally risk free. But there is another risk-free financial instrument, the bank account. If there are two risk-free financial instruments, then they must both earn the same rate of interest. If they didn't, then there would be an arbitrage opportunity: invest in the portfolio with the higher rate of interest by borrowing at the lower. Efficient market theory says that such arbitrage opportunities cannot exist. Thus equating the return on the special delta-hedged portfolio with the bank rate of interest results in an equation, the Black–Scholes equation.

The Black–Scholes equation is a parabolic partial differential equation. Mathematically it is of the same type as the heat or diffusion equation, one of the most widely studied of partial differential equations. Indeed, the diffusion equation has been around for nearly two centuries. Suddenly, finance became a subject of interest to mathematicians; there was more to it than simple compound interest.

4. From theory to practice

The Black–Scholes equation has very simple solutions for the theoretical values of calls and puts. The relevant formulae are in terms of the cumulative distribution function for a normally distributed random variable, and can be interpreted probabilistically. That there were simple formulae for the most important contracts ensured that the Black–Scholes–Merton model would be used by practitioners. To some extent it also made redundant any need for sophisticated numerical techniques for solving partial differential equations.

The model works quite well in practice, but is far from being perfect. The world of finance is far removed from the physical world and there is no reason to believe that there should be any immutable governing laws or principles. Even the idea of 'rational agents', popular in economics, is flawed. It is common experience that human beings are from from rational. But could there ever be an 'irrational behaviour theory'?

Because the model is not perfect, finance researchers and practitioners have invented a number of 'patches' to 'improve' the theory. Often these improvements are completely inconsistent with the rest of the theory and probably serve to distract from the search for what truth there may be.

To summarize the state of finance in the mid-to-late 1970s, there was a well-established theory based on elegant principles that seemed to be not too unsuccessful in practice and, by virtue of simple formulae, was easy to implement.

It is interesting to note that the Black–Scholes formulae for the values of simple options contain a number of easily measured parameters and one not so easily measured parameter. The last is the volatility, the amount of randomness in the evolution of asset prices. This plays a key role in the theoretical pricing of options yet is a parameter that is impossible to accurately observe or measure. This may be seen as a problem with the model. However, almost the opposite became the case; the volatility was seized upon as the perfect quantity for modelling in its own right. Depending on your point of view, this can be seen as either more sophisticated mathematical modelling or as fudging. An unobservable, unmeasurable parameter is the perfect fudge factor.

In the 1980s the fudging began in earnest. More sophisticated models emerged treating volatility as a random variable for example. Simultaneously, the elegant theory of Black, Scholes and Merton was adopted for modelling interest rates and interest rate products, but with little more than a change of notation. There are undoubtedly similarities between the equity and the fixed income markets, and between the dynamics of equity prices and the dynamics of interest rates. But are the similarities so great that a theory that is only so-so for equities can be taken lock, stock and barrel and applied to another market? Well, that is what happened. As the Procter & Gamble story shows, a lot of money rides on the mathematical models, yet common sense and simple statistical analysis seem to have been replaced by a reliance on sophisticated models that not everyone understands.

In the mid 1990s credit risk became the hot topic. Suddenly, emerging markets became the focus of attention, and in emerging markets there is always a real risk of default. Again, the simple principles, model and even formulae of Black, Scholes and Merton were invoked. And with the tiniest of modifications, there was a whole theory of credit risk. This time the modifications were embarrassingly trivial, no more than a change of notation, changing r (for interest rate) to p (for probability of default). Things had gone too far. There was no way that a model based on dynamic hedging (Black–Scholes–Merton) could be applied to the essentially unhedgeable risk of default.

5. The next quarter century

As far as the mathematics is concerned, the last 25 years has almost invariably relied on the following notions.

1. Brownian motion: a mathematical description of diffusion, using random numbers drawn from a normal distribution.
2. Stochastic differential equations: a continuous-time framework for modelling using Brownian motion.
3. Delta hedging: buying/selling contracts that are perfectly correlated with each other.
4. Risk elimination: the ideal result of delta hedging.
5. Correlations between instruments: the price changes in various instruments, such as individual equities, can be related to each other by measurable correlations.
6. Market completeness: perfect delta hedging and risk elimination is possible meaning that options can be artificially created by a suitable buy/sell strategy in the underlying asset. Options are therefore, in a sense, redundant.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

7. Risk-neutral pricing: the possibility of risk elimination means that holders of options should not be rewarded for the extra risk in holding an unhedged option. The return on such a position must be the same as the return on a risk-free investment.

The underlying assumptions in the models, such as the importance of the normal distribution, the elimination of risk, measurable correlations, etc., are incorrect. They are easily shown to be incorrect by relatively simple statistical analysis. The next 25 years will see these foundations being replaced by something less restrictive and therefore more solid. Here are just a few of the new ideas that are starting to appear. The common thread is that they use different mathematics from classical finance with fewer assumptions.

1. Uncertain parameters: as said above, volatility is unobservable and cannot be measured. But clearly the idea of randomness is important in the evolution of asset prices. A recent development in financial modelling has been to treat volatility as uncertain, meaning that we do not specify an exact value but allow it to lie within a specified range. Now instead of a known volatility and a single option value, we have an unknown volatility and a range for option prices. It is natural to think in terms of the worst possible value for the option, what path the volatility must take within its range to give the option its lowest theoretical value. This idea can be applied to any parameters in the classical diffusion model for options (Avellaneda *et al*. 1995; Lyons 1995; Avellaneda & Paras 1996; Wilmott 1998).

2. Crash modelling: undoubtedly, sudden, unhedgeable moves in asset prices or interest rates have an enormous impact on the profitability of investment banks. Often the sudden move is downwards and the bank is a big loser. Yet until recently the only model for such moves was the jump-diffusion model. In that model the effect of the crash was priced into contracts in an 'average' sense. This may be relevant when spread over many market crashes, but if a single crash leads to the collapse of a bank, such a model is clearly irrelevant. Continuing with the theme of worst-case scenarios, recent models have been aimed at determining when is the worst time for a crash, and how big an asset price move leads to the lowest possible portfolio value (Hua & Wilmott 1997). When a portfolio contains options, a fall can even be beneficial. If options can reduce the impact of a crash, what is the best portfolio of options to buy as insurance against a crash?

3. Non-probabilistic value at risk: value at risk has traditionally been an estimate of how much a bank could lose with a given probability. These measures are usually based on the assumption of normally distributed returns. Common sense and experience show that banks collapse for two reasons, mismanagement/poor control of positions and extreme market conditions. We won't worry about the former here, but the latter is just a market crash. As mentioned above it is far more appropriate to examine what is the worst that can happen when trying to determine the future of a bank. During market crashes returns are not normally distributed and the correlations that one measures under day-to-day market conditions are irrelevant; during a crash all correlations become one.

One of the simplest and most popular of the new risk measures has been termed 'CrashMetrics' (Wilmott 1998).

4. Cointegration: cointegration is a far more sensible measure of the relationship between assets than correlation (Alexander & Johnson 1992, 1994). Two assets are cointegrated when, loosely speaking, their time-series do not stray too far from each other. Although this idea is used in asset allocation it has not yet been applied successfully to options markets.

5. Non-probabilistic interest rate modelling: interest rate models are today just equity models with some window dressing. Yet the fixed-income market is different from the equity market in many, many ways, not least of which is the far greater importance of interest rate products. Current approaches are aimed at getting as far away from too precise a modelling as possible. Instead of modelling randomness in a probabilistic sense, as with the uncertain parameter models, worst-case scenarios are examined. Very few assumptions are made about the probabilistic evolution of rates; instead one specifies what is not allowed to happen (Epstein & Wilmott 1998).

6. Utility of credit risk: when there is risk of default in a contract there will almost always be an element of gambling: does the company default or not? Therefore, it is not relevant to model in a risk-neutral framework. Different investors will value the same contract in different ways. A better framework, popular with economists, is that of utility theory. In essence utility is about assigning a value to wealth that captures the idea that an extra £100 means less to a millionaire than to an academic, even though it can be used to purchase exactly the same goods. Utility theory is just beginning to find use in credit risk. There has been some reluctance in the past because of the inevitable nonlinearity introduced into the pricing of contracts and, as a result, sophisticated numerical methods must be employed for solving the governing equations (Ahn *et al*. 1998).

Graduates from many scientific disciplines are joining banks and taking with them a great diversity of knowledge and experience, all of which can only aid the understanding of the financial world. Physicists, engineers and applied mathematicians are joining the traditional MBAs, economists and pure mathematicians. Many universities are now offering masters degrees in quantitative finance; the Masters in (Mathematica/Quantitative) Finance is now seen as a necessary course within universities and is replacing the MBA as a money-spinner.

Mathematical finance is at a turning point. Models of the last 25 years have run their course. The financial world is increasingly sophisticated, profit margins are falling and there have been many well-publicized derivatives fiascos. All of these point at a major change in direction for future models. Over the next quarter century uncertainty will augment randomness in the modelling. Theories will result in price ranges rather than single values. Common sense will return, replacing blind reliance on mathematical models. Market incompleteness will be accepted and no longer feared. This is an exciting time to be a researcher in mathematical finance.

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PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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